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Conducting Spheres in Rectangular Waveguides

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Abstract—**A conducting sphere on the center of the broad wall of a rectangular waveguide causes frequency independent reflections for a wide range of frequencies. This empirically found behavior is confirmed analytically here by perturbational calculations. Furthermore experimental and analytical results are given for a sphere located in the middle of the waveguide cross section.**

I. Introduction

RECENTLY, a problem which seems to be not too complicated when treated perturbationally was reported [1]. This problem has not been solved analytically previously [2].

In [2] the H_{10}-reflection coefficient $|\Gamma|$ of a conductive sphere placed on the center of the broad wall of a rectangular waveguide, as shown in Fig. 1(a), is reported to be

$$|\Gamma|=\frac{5.8(d/b)^3}{1+5.6(d/b)^4} \tag{1}$$

where d = the sphere diameter and b = narrow waveguide wall. This behavior had been found empirically and is independent of frequency over a wide range of frequencies, thus providing a convenient means for wide-band impedance matching.

Before treating this problem we will assume the sphere to be located, not adjacent to a wall, but in the middle of the cross section as shown in Fig. 1(b). Both arrangements will be represented by the equivalent circuit type of Fig. 2, consisting of a symmetrical T-circuit within a transmission line of real wave impedance Z equal to that of the waveguide. The sphere is assumed to be a perfect conductor; so the elements of the T-circuit will be nonresistive.

To determine X and B use is made of the formula [3, p. 4]

$$\frac{\omega-\omega_0}{\omega}=-\frac{\iiint(\Delta\epsilon \boldsymbol{E}\cdot\boldsymbol{E}_0^*+\Delta\mu \boldsymbol{H}\cdot\boldsymbol{H}_0^*)dV}{\iiint(\epsilon \boldsymbol{E}\cdot\boldsymbol{E}_0^*+\mu \boldsymbol{H}\cdot\boldsymbol{H}_0^*)dV} \tag{2}$$

which gives the relative resonant frequency deviation $(\omega-\omega_0)/\omega$ of a conducting cavity caused by inserting a small piece of material with permittivity $\epsilon=\epsilon_0+\Delta\epsilon$ and permeability $\mu=\mu_0+\Delta\mu$ into the formerly empty cavity (μ_0, ϵ_0). $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are the complex amplitudes of the electric and magnetic fields in the empty cavity, $\boldsymbol{E}_0^*$ and $\boldsymbol{H}_0^*$ their complex conjugates, and $\boldsymbol{E}$ and $\boldsymbol{H}$ the fields in the perurbed cavity. The boldface letters denote vectors; and the integral in the numerator is effectively extended only over those regions of the cavity volume in which $\Delta\epsilon$ and $\Delta\mu$ are not equal to zero.

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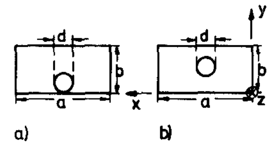

Fig. 1. Conducting sphere in a rectangular waveguide. (a) On the center of the broad wall. (b) In the middle of the cross section.

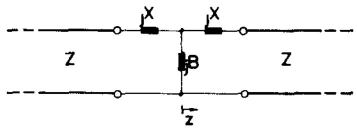

Fig. 2. Equivalent circuit of a sphere within a rectangular waveguide

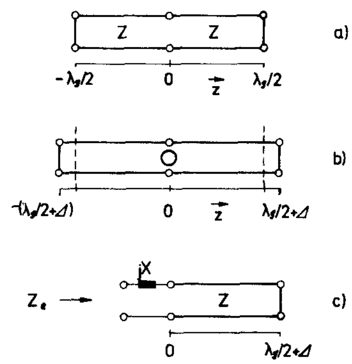

Fig. 3. H_{102}-resonator. (a) Empty cavity. (b) Cavity with conducting sphere. (c) Equivalent circuit of half the cavity.

Generally, $\boldsymbol{E}$ and $\boldsymbol{H}$ are unknown, but for small inserts of certain shapes they may be calculated from the static approximation using the depolarizing factor or demagnetizing factor, respectively. Furthermore (2) may be applied to inserts of ideal conductivity by choosing $\epsilon\to\infty$ and $\mu\to 0$, [3, p. 9].

II. Series Reactance

In order to determine the series reactance X the empty waveguide is assumed to be short circuited at the ends, forming an H_{102}-resonator, of guide wavelength λ_g as shown by the equivalent circuit in Fig. 3(a). From the z component Ψ of the electric vector potential

$$\boldsymbol{F}=-\operatorname{rot}\boldsymbol{E} \tag{3}$$

with

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$$\Psi_{102} = D\cos\frac{\pi x}{a}\sin\frac{2\pi z}{\lambda_g} \tag{4}$$

and the arbitrary factor D and from making use of Maxwells equations

$$\mathrm{rot}\,\boldsymbol{H} = j\omega\epsilon_0\boldsymbol{E} \tag{5}$$

$$-\mathrm{rot}\,\boldsymbol{E} = j\omega\mu_0\boldsymbol{H} \tag{6}$$

and the separation condition

$$k^2 = \omega^2\mu_0\epsilon_0 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{\lambda_g}\right)^2 \tag{7}$$

the only nonzero components of the fields in the resonator follow as

$$E_{y0} = -D\frac{\pi}{a}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi z}{\lambda_g}\right) \tag{8}$$

$$H_{x0} = -\frac{D}{j\omega\mu}\frac{2\pi^2}{a\lambda_g}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi z}{\lambda_g}\right) \tag{9}$$

$$H_{z0} = \frac{D}{j\omega\mu}\frac{\pi^2}{a^2}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi z}{\lambda_g}\right). \tag{10}$$

When the small sphere is placed into the middle of the resonator, the resonant frequency will be changed. But this change can be canceled by symmetrically prolonging the resonator by two small lengths Δ as shown in Fig. 3(b). Now, with zero resonant frequency shift, the numerator in (2) must be zero. With the only appreciable field component H_{x0} at the discontinuities, being nearly homogenous at the sphere and being parallel to the plane at the shifted walls, the static approximations

$$H_x = \frac{3}{2+\mu/\mu_0}H_{x0} \tag{11}$$

at the sphere and

$$H_x = H_{x0} \tag{12}$$

at the shifted walls lead to

$$0 = \frac{3}{2+\mu/\mu_0}V_{\text{sphere}} - 2\Delta b\int_0^a \sin^2\frac{\pi x}{a}\,dx. \tag{13}$$

With the volume $V_{\text{sphere}} = \pi d^3/6$, the value of the integral being a/2, and $\mu = 0$ we obtain

$$\Delta = \frac{\pi d^3}{4ab}. \tag{14}$$

In the symmetry plane $z=0$, $E_y = 0$. That means the input impedance Z_e of half the resonator, as shown in Fig. 3(c), must be zero. With the input impedance of the short circuited transmission line of length $\lambda_g/2+\Delta$ and phase constant $\beta = 2\pi/\lambda_g$ [4]

$$Z_k = jZ\tan[\beta(\lambda_g/2+\Delta)] \tag{15}$$

it follows that

$$0 = Z_e = jX + jZ\tan(\pi + \beta\Delta) \tag{16}$$

or

$$\frac{X}{Z} = -\tan\left(\beta\frac{\pi d^3}{4ab}\right). \tag{17}$$

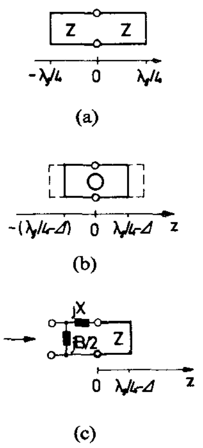

Fig. 4. H_{101}-resonator. (a) Empty cavity. (b) Cavity with conducting sphere. (c) Equivalent circuit of half the cavity.

III. Shunt Susceptance

In order to obtain the shunt susceptance the waveguide is now assumed to be made a H_{101}-resonator by shorting it as shown in Fig. 4(a). With an arbitrary constant G the fields within it, corresponding to the former part, may then be described by

$$\Psi_{101} = G\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi z}{\lambda_g}\right) \tag{18}$$

$$E_{y0} = -G\frac{\pi}{a}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi z}{\lambda_g}\right) \tag{19}$$

$$H_{x0} = \frac{G}{j\omega\mu}\frac{2\pi^2}{a\lambda_g}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi z}{\lambda_g}\right) \tag{20}$$

$$H_{z0} = \frac{G}{j\omega\mu}\frac{\pi^2}{a^2}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi z}{\lambda_g}\right). \tag{21}$$

Now, corresponding to Fig. 4(b), the sphere is located at the middle of the resonator, where the appreciable field is now an electrical one, see (19) to (21). The change in resonant frequency will be canceled again by shifting the short circuiting walls in the regions of only appreciable magnetic field.

The static approximations for the fields are now

$$E_y = \frac{3}{2+\epsilon/\epsilon_0}E_{y0} \tag{22}$$

within the sphere and

$$H_x = H_{x0} \tag{23}$$

at the shifted walls. The zero numerator of (2) leads to

$$0 = \frac{3(\epsilon-\epsilon_0)}{2+\epsilon/\epsilon_0}\frac{1}{\pi a}V_{\text{sphere}} + (\mu-\mu_0)\frac{4\pi b\Delta}{\omega^2\mu_0^2\lambda_g^2}. \tag{24}$$

With V_{sphere} being as before, $k = \omega\sqrt{\mu_0\epsilon_0}$, $\epsilon \to \infty$ and $\mu = 0$ one obtains

$$\Delta = \frac{d^3\lambda_g^2 k^2}{8\pi ab}. \tag{25}$$

Due to of symmetry, at $z=0, H_x = 0$ so that the input admittance Y_e in Fig. 4(c) must vanish, leading to

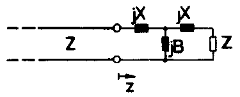

Fig. 5. Equivalent circuit to calculate the reflection coefficient of the sphere.

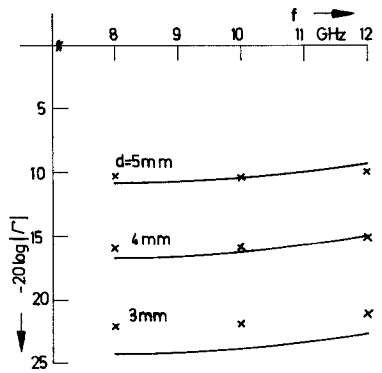

Fig. 6. Reflection coefficient of a sphere in the middle of the cross section in an X-band waveguide (a=22.86 mm, b=10.16mm). —— follows from (31). ××× measurements.

$$\frac{B}{2}=\frac{1}{X+Z\tan[\beta(\lambda_g/4-\Delta)]} \tag{26}$$

or approximately

$$BZ\approx 2\tan(\beta\Delta)=2\tan\left(\frac{\pi d^3k^2}{2\beta ab}\right). \tag{27}$$

IV. Reflection Coefficient

The reflection coefficient of the sphere may now be calculated from the equivalent circuit in Fig. 5; a network of input impedance

$$Z_r=Z\frac{1+2j(X/Z)-(X/Z)BZ(1+jX/Z)}{1+jBZ(1+jX/Z)} \tag{28}$$

is connected to the transmission line of wave impedance Z. The reflection coefficient $\Gamma=(Z_r/Z-1)/(Z_r/Z+1)$ is then approximately, i.e., for X/Z and BZ being much smaller than one,

$$\Gamma\approx -j\left[\frac{1}{2}BZ-\frac{X}{Z}\right]. \tag{29}$$

Using (17) and (27) with the tangent-functions replaced by their arguments, with $k_c=\pi/a$ and with

$$\beta=k\sqrt{1-\left(\frac{k_c}{k}\right)^2} \tag{30}$$

one obtains

$$\Gamma\approx -j\left(\frac{d}{b}\right)^3\left(\frac{\pi b}{2a}\right)^2\frac{k}{k_c}\left[\frac{2}{\sqrt{1-(k_c/k)^2}}+\sqrt{1-(k_c/k)^2}\right]. \tag{31}$$

The magnitude of this reflection coefficient has been confirmed by measurements with steel balls held by adhesive tapes in the middle of a slotted X-band waveguide. Results are shown in Fig. 6. The deviations occurring with the ball of $d=3$ mm are assumed to result from additional small reflections caused by the adhesive tapes.

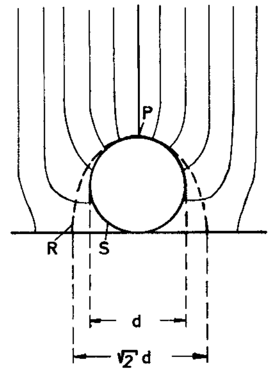

Fig. 7. Estimated configuration of the electric field near the sphere S on the conducting ground and half the ellipsoid of rotation R with equal curvature radius at point P.

V. Sphere Adjacent to the Wall

Now the conducting sphere is located on the bottom $y=0$ as shown in Fig. 1(a). It will be assumed that due to the nearness of the electrically conducting wall the field distribution around the sphere will be significantly changed only when there is an E_z-field; the change in the H_x-field distribution will be neglected. Then only the value of B will be altered.

The expected electric field distribution around the conducting sphere S on the conducting ground is sketched in Fig. 7. The field strength will be low in the shade between sphere and ground. Therefore, this almost field free region may be filled with a conductor without effecting the result. So the sphere is now replaced by half a conducting ellipsoid of rotation R with equal curvature radius $d/2$ at point P and of half the long axis being equal to d. The short axis is then determined by $2\cdot d$ [5, p. 177].

Such a shading effect will not appear in the case of the H_x-field; therefore it is assumed in the following that the magnetic field distribution is not effected by the conducting ground. The demagnetizing factor of the ellipsoid R may be interpolated from [6, p. 22] yielding $N=0.245$ and because of duality used as the depolarizing factor in the E_z-case, yielding

$$E_y=\frac{1}{1+N(\epsilon-\epsilon_0)/\epsilon_0}E_{y0}=\frac{4.08}{3.08+\epsilon/\epsilon_0}E_{y0} \tag{32}$$

instead of (22). Furthermore in (24) V_{sphere} has to be replaced by the volume of half the ellipsoid, being

$$V_{\text{ell}}/2=\frac{\pi}{3}d^3$$

twice that of the sphere.

This results in a reflection coefficient whose magnitude is

$$|\Gamma|=\left(\frac{d}{b}\right)^3\left(\frac{\pi b}{2a}\right)^2\frac{k}{k_c}\left[\frac{5.44}{\sqrt{1-(k_c/k)^2}}+\sqrt{1-(k_c/k)^2}\right]. \tag{33}$$

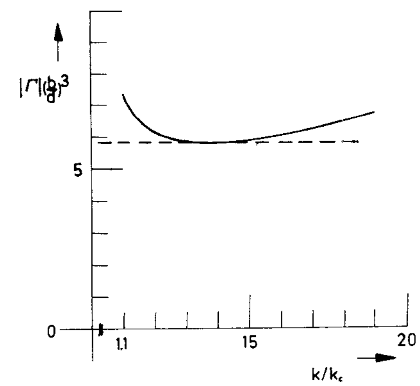

Fig. 8. Reflection coefficient of a sphere on the center of the broad wall in a rectangular waveguide with $b=10.16$ mm and $b/a=0.445$. ——theoretically found behavior as given by (33) - - - - empirically found behavior for small sphere diameters, see [2].

This coefficient, normalized to $(d/b)^3$, versus normalized frequency k/k_c is shown in Fig. 8. It agrees quite well with the small diameter approximation of (1),

$$|\Gamma| = 5.8(d/b)^3 \tag{34}$$

which was found experimentally. Furthermore, the frequency dependence of the reflection coefficient was reported to be within 10 percent of its midband value between $1.22 \leqslant k/k_c \leqslant 1.7$, [2], which is also confirmed by the analytical calculations leading to (33) and Fig. 8.

VI. Conclusions

A method has been shown to calculate the T-section equivalent circuit and the reflection coefficient for a metallic sphere in a rectangular waveguide. Experimental and theoretical results agree quite well. The theoretical method applied here can also be applied to the T-section equivalent circuits for obstacles with other forms in uniform waveguides.

Acknowledgment

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References

[1] P. I. Somlo, Talk at Physikalisch-Technische Bundesanstalt, Braunschweig (PTB), July 1978.
[2] ——, "Conductive contacting spheres on the centre of the broad wall of rectangular waveguides," *Electron. Lett.*, vol. 8, no. 20, 1972.
[3] H.-G. Unger, *Elektromagnetische Wellen II*. Braunschweig, Germany: Vieweg, 1967.
[4] ——, *Theorie der Leitungen*. Braunschweig, Germany: Vieweg, 1967.
[5] I. N. Bronstein and K. A. Semendjajew, *Taschenbuch der Mathematik*, 7th ed. Zürich/Frankfurt am Main, Germany: Harri Deutsch, 1967.
[6] S. Chikazumi, *Physics of Magnetism*. New York: Wiley, 1964.

On the Propagation of Leaky Waves in a Longitudinally Slotted Rectangular Waveguide

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***Abstract*—The field theory approach is used to study leaky-wave propagation in a rectangular waveguide with long nonresonant slots in the narrow walls. Radiation from the slots is confined by parallel plates which act as transmission lines guiding the energy away from the slots. The complex dispersion equations for TE waves are examined and solved using an iterative numerical technique. Propagation characteristics both in the axial and transverse directions are presented, along with the electric field distribution and power flow. Restrictions on the analysis and on the power-handling capacity imposed by slot width also are described. Measurements of the phase characteristics of the dominant mode are in good agreement with theoretical values.**

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I. Introduction

SLOTTED WAVEGUIDES are used both as applicators for material processing [1], [2] and as antennas [3]. In particular, traveling-wave slotted structures are used in antenna design [4] because of their ease of construction and their ability to control radiation by varying the slot geometry along the length of the guide.

Typically, the analysis of leaky-wave structures has been carried out using a microwave network representation of the transverse discontinuity [5]. This requires a previous knowledge of the fields which are regarded as weak perturbations of those which would exist in the closed perfectly conducting guide. An alternative approach is to determine the propagation coefficient from the field solution which satisfies the boundary conditions.

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